

A GEOMETRIC INTERPRETATION OF THE TRIANGULATED STRUCTURE OF m -CLUSTER CATEGORIES

LAMBERTI LISA

ABSTRACT. The aim of this note is to answer several open problems arising from the geometric description of the m -cluster categories of type A_n and their realization in terms of the m -th power of a translation quiver. In particular, we give a geometric interpretation of the triangulated structure of m -cluster categories. Furthermore, we characterize all the connected components arising from a cluster category when taking the m -th power of its Auslander-Reiten quiver.

1. INTRODUCTION

Cluster categories were introduced in [CCS] and [BMRRT] with the aim of modeling the features of cluster algebras, discovered by S. Fomin and A. Zelevinsky in [FZ]. In [BMRRT] they were defined as orbit categories of the bounded derived category of finitely generated modules over a finite dimensional hereditary algebra H over a field k by a certain auto-equivalence. A number of cluster categories are approachable in a more concrete way. In fact, in [CCS] the cluster categories associated to the path algebras $H := kQ$ over a Dynkin quiver of type A_n were expressed in geometric terms by means of triangulations of a certain polygon Π . The objects of the cluster category of [CCS] are direct sums of diagonals and the morphisms are given by certain rotations around vertices of Π modulo the so-called mesh relations which allow the exchange of certain types of rotations. Similarly, R. Schiffler provided in [Sc] a geometric description of cluster categories of type D_n by means of polygons which have a single puncture in their centers. More geometric models have been given in [BaM3], [BZ] and [To].

Cluster categories have a number of interesting properties: they are triangulated, the projection functor from $\mathcal{D}^b(\text{mod } H)$ to the cluster category is a triangle-functor and they are Calabi-Yau of dimension 2. This was all proven by B. Keller in [Ke1]. Furthermore, these categories are Krull-Schmidt and have Auslander-Reiten triangles [BMRRT]. In addition, the geometric model is beautifully linked to cluster algebras: in type A_n we have the bijection between the isomorphism classes of indecomposable objects of $\mathcal{C} := \mathcal{C}_{A_n}$ and the diagonals of Π as well as between the diagonals of Π and cluster variables of a cluster algebra of type A_n . Under the latter bijection maximal sets of non-crossing diagonals correspond to clusters, and mutations of cluster variables can be interpreted as flips of diagonals.

As a generalization of cluster categories, B. Keller introduced in [Ke1] the so called m -cluster categories \mathcal{C}^m , for $m \in \mathbb{N}$. These categories, studied for example by [BaM1], [BaM2], [Jø], [Th], [To], [Wr], [Zh], [ZZ], have the same properties as the original ones, except that this time they are Calabi-Yau of dimension $m + 1$ [Ke1]. Also for certain m -cluster categories a geometric description was provided. In [BaM1], K. Baur and R. Marsh expressed the m -cluster categories of type A_n by means of the so-called m -diagonals which are the diagonals dividing a given

Key words and phrases. m -cluster categories, triangulated structure, Auslander-Reiten triangles, orbit categories, Auslander Reiten quivers, m -th power of a translation quiver.

$(mn + 2)$ -gon Π into two smaller polygons whose number of sides are congruent to 2 modulo m . This construction recovers the original one given in [CCS] for $m = 1$. In [BaM2] the authors gave a geometric description of m -cluster categories of type D_n generalizing Schiffler's work [Sc] for cluster category for type D_n .

In [BaM1], K. Baur and R. Marsh establish a combinatorial method enabling one to recover important information about \mathcal{C}^m starting from \mathcal{C} . This information is recorded in the Auslander-Reiten quivers of the corresponding categories. In fact they showed that one can obtain the Auslander-Reiten quiver of an m -cluster category of type A_{n-1} , starting from the one of a cluster category of type A_{nm-1} . For this, [BaM1] introduced the concept of the m -th power of a stable translation quiver, Γ , obtained from the set of diagonals of the $nm + 2$ -gon Π , with arrows arising from paths in Γ . In general, the m -th power of Γ consists of several connected components. One of them is isomorphic to the Auslander-Reiten quiver of an m -cluster category of type A_{n-1} , as K. Baur and R. Marsh showed.

Concerning the content of this note, our interest is twofold. On one hand, we are interested in describing the triangulated structure of the m -cluster categories of type A_{n-1} on the level of the geometric model, and understanding it through the concept of the m -th power of Γ . On the other hand, we will give a complete characterization of all the connected components arising with this procedure.

More precisely, the plan is as follows. We start by recalling the necessary definitions and preliminary results, then we present the m -cluster categories of type A_{n-1} and show how one can obtain their Auslander-Reiten quivers in geometric terms. We will also recall the concept of the m -th power of a translation quiver (Section 2). In the third section we explain how the triangulated structure of the m -cluster category can be recovered from the polygon Π . For this we will first study Auslander-Reiten triangles of \mathcal{C}^m which turn out to have a beautiful geometric description. Secondly, we will explain the link between Auslander-Reiten triangles of \mathcal{C}^m and \mathcal{C} . Here, we view the former category as obtained from the latter by taking the m -th power of its Auslander Reiten quivers. Finally, we study arbitrary triangles coming from a morphism μ between two objects in \mathcal{C}^m and provide a geometric description of the third object of the triangle of μ , i.e. of $\text{Cone}(\mu) \in \mathcal{C}^m$. Combining these results will provide a geometric understanding of the triangulated structure of all orbit categories arising from \mathcal{C} through the m -th power of the Auslander-Reiten quiver of \mathcal{C} . A precise description of the different connected components arising will be given in Section 4. In the last section we illustrate on two examples the results of Section 4.

Acknowledgments: I would like to thank Prof. K. Baur for introducing me to such an interesting field, for the inspiring discussions we had and for her time in reading my drafts.

2. DEFINITIONS AND PRELIMINARIES

In this section we introduce some terminology following mainly [BaM1], [BMRRT], [CCS] and [Ke1].

Throughout this paper let $n, m \in \mathbb{N}$ and fix an $N := nm + 2$ -sided regular polygon Π with vertices numbered clockwise. Consider all operations on the vertices of Π modulo $nm + 2$. Unless stated otherwise Q will always be a Dynkin quiver of type A_{n-1} and $\mathcal{D}^b(\text{mod } kQ)$ the bounded derived category of finite-dimensional modules over the path algebra kQ , where k is an algebraically closed field.

The m -cluster category of type A_{n-1} is defined as the orbit category of the bounded derived category under the action of the cyclic group generated by the auto-equivalence $\tau^{-1}\Sigma^m$:

$$\mathcal{C}^m := \mathcal{D}^b(\text{mod } kQ) / (\tau^{-1}\Sigma^m)^{\mathbb{Z}},$$

where τ denotes the Auslander-Reiten translation and Σ^m is the composition of the shift functor Σ with itself m times. The isoclasses of objects of \mathcal{C}^m are the $\tau^{-1}\Sigma^m$ -orbits $\tilde{X} := ((\tau^{-1}\Sigma^m)^i X)_{i \in \mathbb{Z}}$ of objects $X \in \mathcal{D}^b(\text{mod } kQ)$. For two objects \tilde{X}, \tilde{Y} in \mathcal{C}^m we have:

$$\text{Hom}_{\mathcal{C}^m}(\tilde{X}, \tilde{Y}) = \bigoplus_{i \in \mathbb{Z}} \text{Hom}_{\mathcal{D}^b(\text{mod } kQ)}((\tau^{-1}\Sigma^m)^i X, Y).$$

For simplicity later on we will omit the tilde and indicate a representative of the preimage of an object $X \in \mathcal{C}^m$ under the projection functor $\pi : \mathcal{D}^b(\text{mod } kQ) \rightarrow \mathcal{C}^m$ by $\pi^{-1}(X)$. This category is k -linear triangulated ([Ke1]), Krull-Schmidt and has Auslander Reiten triangles ([BMRRT]). We recall that being Krull-Schmidt means that each object decomposes uniquely up to isomorphism and permutation of factors into a finite sum of indecomposable objects. The latter are objects which are neither zero nor a direct sum of two non-zero objects. Furthermore, remark that the shift functor Σ and the Auslander-Reiten translate τ_m in \mathcal{C}^m are induced by the ones in $\mathcal{D}^b(\text{mod } kQ)$. For $m = 1$ we omit the indices in \mathcal{C}^m and τ_m and simply write \mathcal{C}, τ .

Recall that the *Auslander-Reiten quiver* of a Krull-Schmidt category \mathcal{K} is a quiver whose vertices are the isomorphism classes of indecomposable objects of \mathcal{K} and where the number of arrows between two vertices $[X]$ and $[Y]$ is given by the dimension of the space of *irreducible morphisms* between the indecomposables X and Y :

$$\text{Irr}_{\mathcal{K}}(X, Y) := \text{rad}_{\mathcal{K}}(X, Y) / \text{rad}_{\mathcal{K}}^2(X, Y).$$

Here $\text{rad}_{\mathcal{K}}(X, Y)$ is the subspace of $\text{Hom}_{\mathcal{K}}(X, Y)$ formed by all non isomorphisms, and $\text{rad}_{\mathcal{K}}^2(X, Y)$ is the subspace of all non isomorphisms admitting a non trivial factorization.

To Q we can associate also another quiver which we denote by $\mathbb{Z}Q$. Its vertices are labeled by pairs (n, i) where $n \in \mathbb{Z}$ and i is a vertex of Q , and if there is an arrow in Q from i to j then we draw an arrow from (n, i) to (n, j) and one from (n, j) to $(n+1, i)$. In this way $\mathbb{Z}Q$ is an infinite strip of copies of Q . Furthermore, one can define a translation map τ on $\mathbb{Z}Q$ by sending (n, j) to $(n-1, j)$, this makes $\mathbb{Z}Q$ a stable translation quiver as defined by C. Riedtmann, [Ri].

In particular, D. Happel showed in [Ha2] that the Auslander-Reiten quiver of $\mathcal{D}^b(\text{mod } kQ)$ is isomorphic to $\mathbb{Z}Q$. From this result it follows that $\mathcal{D}^b(\text{mod } kQ)$ is independent of the orientation of Q , see Section 4.8 in [Ha2]. Furthermore, the Auslander-Reiten quiver of \mathcal{C}^m is isomorphic to the quotient $\mathbb{Z}Q / \varphi^m$, see Proposition 1.3 in [BMRRT] where φ^m is the graph automorphism induced by the auto-equivalence $\tau^{-1}\Sigma^m$. For simplicity in the following we sometimes write $\tau^{-1}\Sigma^m$ instead of φ^m .

After having reminded the algebraic definition of m -cluster categories we will recall the geometric approach following [CCS] and [BaM1]. In this case the Auslander-Reiten quiver of \mathcal{C}^m admits another description, as follows from the next theorem due to P. Caldero, F. Chapoton and R. Schiffler (Section 2 in [CCS]) for the case $m = 1$ and to K. Baur and R. Marsh (Proposition 5.5 in [BaM1]) for $m \geq 1$.

Let Π be as before. As usual we refer to a *diagonal* of Π as a straight line between two non-adjacent vertices of Π . Whereas an *m-diagonal* in Π is a diagonal which divides Π into an $(mj+2)$ -gon and an $(m(n-j)+2)$ -gon where $j = 1, \dots, \lceil \frac{n-1}{2} \rceil := \min\{l \in \mathbb{Z} | l \geq \frac{n-1}{2}\}$. To the m -diagonals of Π one can associate a quiver $\Gamma_{A_{n-1}}^m$

as follows: its vertices are m -diagonals and the arrows send the diagonal (i, j) to $(i, j+m)$ and $(i+m, j)$ whenever the image is also an m -diagonal. If $m = 1$, we omit the index m and write $\Gamma_{A_{n-1}}$. Furthermore, defining an automorphism τ_m as the map sending the m -diagonal (i, j) to $(i-m, j-m)$ one makes the pair $(\Gamma_{A_{n-1}}^m, \tau_m)$ into a (stable) translation quiver in the sense of C. Riedtmann. As τ_m is defined on all vertices, the translation quiver $(\Gamma_{A_{n-1}}^m, \tau_m)$ is stable. We refer to the examples in Section 5 for an illustration of this construction.

Theorem 2.1. *Let Q be a quiver of Dynkin type A_{n-1} , and let $m \geq 1$. Then the Auslander-Reiten quiver of \mathcal{C}^m is isomorphic to $\Gamma_{A_{n-1}}^m$.*

Because of this theorem in the following we tacitly switch from one quiver to the other.

Now we recall the procedure of taking the m -th power of a given translation quiver.

Definition 2.2. *Let (Γ, τ) be a translation quiver. The quiver $(\Gamma)^m$ whose vertices are the same as the ones from Γ and whose arrows are sectional paths of length m is called the m -th power of Γ . A path $(x = x_0 \rightarrow x_1 \rightarrow \dots \rightarrow x_{m-1} \rightarrow x_m = y)$ in Γ is said to be sectional if $\tau x_{i+1} \neq x_{i-1}$ for $i = 1, \dots, m-1$ (for which τx_{i+1} is defined).*

Observe that the pair $((\Gamma)^m, \tau^m)$ where $\tau^m := \tau \circ \dots \circ \tau$, composed m times, is again a translation quiver (Theorem 6.1 in [BaM1]), and if Γ is a stable translation quiver, then so is $((\Gamma)^m, \tau^m)$. Furthermore, the next result (Proposition 7.1 in [BaM1]) shows how the quivers introduced so far are related to each other.

Proposition 2.3. *$(\Gamma_{A_{n-1}}^m, \tau_m)$ is a connected component of the m -th power of $(\Gamma_{A_{nm-1}}, \tau)$.*

Generally the m -th power of $(\Gamma_{A_{nm-1}}, \tau)$ decomposes into several different connected components. In the last section we will characterize all of them and we will see that apart from $(\Gamma_{A_{n-1}}^m, \tau_m)$ there may also occur other components which are isomorphic to the Auslander-Reiten quiver of \tilde{m} -cluster categories of type A_n , for suitable $\tilde{m} \geq 1$.

3. GEOMETRIC DESCRIPTION OF THE TRIANGULATED STRUCTURE OF m -CLUSTER CATEGORIES

In this section we explain how the triangulated structure of an m -cluster category of type A_{n-1} can be comprehended geometrically, and how it can be understood through the combinatorial tool of taking sectional paths of length m . The first part is devoted to the study of Auslander-Reiten (AR) triangles, also called almost split triangles, which have a nice geometric description as we will see. In the second part we will give the link between AR triangles of \mathcal{C}^m and the ones of \mathcal{C} . Here we use that m -cluster category of type A_{n-1} can be obtained from the cluster category of type A_{nm-1} by taking the m -th power of the (translation) quiver $(\Gamma_{A_{nm-1}}, \tau)$. In the last part we describe how the triangles $A \xrightarrow{\mu} B \rightarrow C \rightarrow \Sigma A$ in \mathcal{C} and \mathcal{C}^m associated to an arbitrary morphism $\mu : A \rightarrow B$ are linked. For this we will concretely determine the geometric characterization of the object C .

In the following we simply write $\mathcal{D} := \mathcal{D}^b(\text{mod } kQ)$, since the choice of Q should be clear from the context. Recall that the indecomposables of \mathcal{D} are isomorphic to the stalk complexes with indecomposable stalk (see Paragraph I.5. in [Ha2]). That is: complexes X^\bullet for which there is an index i_0 such that $X^{i_0} \neq 0$ and $X^i = 0$ for all $i \neq i_0$. The object X^{i_0} is then called stalk. This justifies the identification of $\text{Ind}(\mathcal{D})$ with $\text{Ind}(\text{mod } kQ)$ used in the following, where $\text{Ind}(\cdot)$ denotes the full subcategory of indecomposable objects of the corresponding category.

3.1. Auslander-Reiten triangles of \mathcal{C}^m . For the geometric description of the AR triangles it is convenient to introduce the following and observe that almost split sequences are uniquely determined up to isomorphisms by their ending or starting terms.

Notice that we always take the endpoints of diagonals modulo $N := nm + 2$.

Lemma 3.1. *The m -diagonals (i, j) and $(i - m, j - m)$ cross inside Π for all (i, j) .*

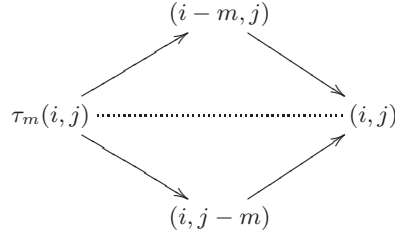
Proof. A m -diagonal (i, j) can be written as $(i, i + km + 1)$ where we assume $i < j$ when writing (i, j) . From the definition of τ_m one deduces that the rotated copy (i', j') of (i, j) can be written as $(i - m, i + (k - 1)m + 1)$. Then, one of the following inequalities holds:

$$i' < i < j' < j, \quad \text{or} \quad i < i' < j < j',$$

depending on whether $j < j'$ or $j' < j$, and these inequalities define a crossing inside Π . \square

Definition 3.2. *We call the set of m -diagonals $(i, j), (i, j - m), (i - m, j), \tau_m(i, j) = (i - m, j - m)$ written modulo N the framed set of m -diagonals of (i, j) . The frame of a crossing $(i, j), (i - m, j - m)$ is given by the m -diagonals $(i, j - m), (i - m, j)$.*

Notice that framed sets of m -diagonals arise from meshes in the translation quiver



defined by 4 or 3 vertices depending on where (i, j) lies. Furthermore, the objects corresponding to the tuples $(i - m, j)$ or $(i, j - m)$ which are not diagonals in Π are viewed as zero. In Figure 1 two framed sets of m -diagonals are represented.

Next we point out a link which follows from the definition of AR quivers and Theorem 2.1, and which will often be used in the following.

Remark 3.3. *The almost split sequence*

$$0 \rightarrow M_{\tau_m(i, j)} \rightarrow M_{(i-m, j)} \oplus M_{(i, j-m)} \rightarrow M_{(i, j)} \rightarrow 0$$

ending at $M_{(i, j)}$ corresponds to the framed sets of m -diagonals of (i, j) .

For 1-cluster categories the description of almost split sequences in $\text{mod } kQ$ for Q of type A_n is mentioned in the proof of theorem 5.1 in [CCS]. For type D_n , R. Schiffler gave the geometric interpretation of AR triangles in [Sc].

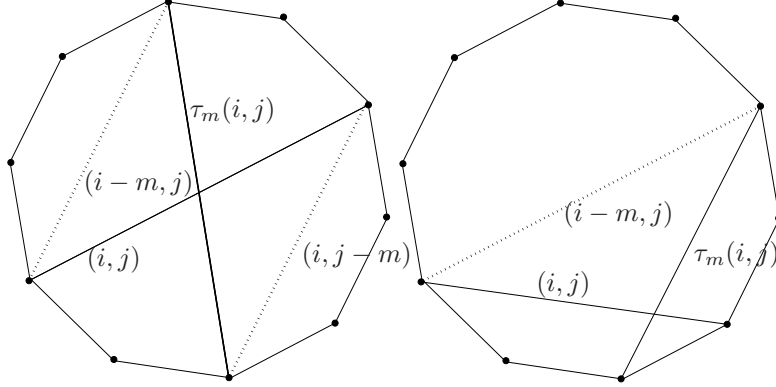
After the previous observation the description of AR triangles of \mathcal{C}^m also becomes clear. Recall that $\pi : \mathcal{D} \rightarrow \mathcal{C}^m$ is the projection functor is also clear.

Proposition 3.4. *Let $M_{(i, j)} \in \text{Ind}(\mathcal{C}^m)$. Then its AR triangle is*

$$M_{\tau_m(i, j)} \rightarrow M_{(i-m, j)} \oplus M_{(i, j-m)} \rightarrow M_{(i, j)} \xrightarrow{\pi(w)} \Sigma M_{\tau_m(i, j)},$$

with w different from zero.

Proof. We divide the proof into two cases depending on whether the indecomposable object $M_{(i, j)}$ corresponds to a projective object of $\text{mod } kQ$ or not. Assume first it does not correspond to a projective. Then, since the category $\text{mod } kQ$ is hereditary,

FIGURE 1. Framed set of m -diagonals

the almost split sequence ending in $M_{(i,j)}$ described in Lemma 3.3 can be embedded into the following AR triangle of \mathcal{D} :

$$M_{\tau_m(i,j)} \rightarrow M_{(i-m,j)} \oplus M_{(i,j-m)} \rightarrow M_{(i,j)} \xrightarrow{w} \Sigma M_{\tau_m(i,j)},$$

with $w \in \text{Hom}_{\mathcal{D}}(M_{(i,j)}, \Sigma M_{\tau_m(i,j)}) = \text{Ext}_{\text{mod } kQ}^1(M_{(i,j)}, M_{\tau_m(i,j)})$, see I.4.7 in [Ha2]. Furthermore, as we saw in Lemma 3.1 the diagonals (i, j) and $\tau_m(i, j)$, which correspond to $M_{(i,j)}$ and $M_{\tau_m(i,j)}$, cross. Hence

$$\text{Ext}_{\text{mod } kQ}^1(M_{(i,j)}, M_{\tau_m(i,j)}) \neq 0,$$

as it follows from Remark 5.3 in [CCS]. Applying the triangle functor $\pi : \mathcal{D} \rightarrow \mathcal{C}^m$ to it one gets the described triangle in \mathcal{C}^m corresponding to the mesh ending in the vertex (i, j) of $\Gamma_{A_{n-1}}^m$. For indecomposable objects of $\text{mod } kQ$ one has that the AR triangle, in this case also called connecting triangle, is the claimed one by Lemma 4.3 in [Ha1]. \square

In the following we will refer to the triangle of Proposition 3.4 as the *AR triangle* of $M_{(i,j)}$. Notice that with the same arguments as the one described in the proof of the previous lemma and taking $Q = A_{nm-1}$ one obtains the AR triangle of $M_{(i,j)}$ in \mathcal{C} associated to the vertex (i, j) of $(\Gamma_{A_{nm-1}}, \tau)$.

In geometric terms the previous result can be restated. Observe that the automorphism τ_m corresponds to an anticlockwise rotation of $\rho_m := \frac{2m\pi}{N}$ degrees inside Π around the center.

Corollary 3.5. *The AR triangle of $M_{(i,j)}$ in \mathcal{C}^m*

$$M_{\tau_m(i,j)} \rightarrow M_{(i-m,j)} \oplus M_{(i,j-m)} \rightarrow M_{(i,j)} \xrightarrow{w} \Sigma M_{\tau_m(i,j)},$$

is described by union of the framed set of m -diagonals of (i, j) with an anticlockwise rotation of (i, j) around the center of $2\rho_m$ degrees.

Proof. Follows from the two previous results and the fact that the shift functor Σ in \mathcal{C}^m is induced from the shift functor in \mathcal{D} and satisfies $\Sigma \cong \tau_m$. Thus it corresponds to an anticlockwise rotation of ρ_m degrees around the center of Π , and the last term in the AR triangle of $M_{(i,j)}$ in \mathcal{C}^m satisfies: $\Sigma M_{\tau_m(i,j)} = M_{2\tau_m(i,j)}$, i.e. is obtained from a rotation of (i, j) of $2\rho_m$ degrees. \square

We saw in Proposition 2.3 how the category \mathcal{C}^m is related to \mathcal{C} through the procedure of the m -th power of the translation quiver $(\Gamma_{A_{nm-1}}, \tau)$. Now, we will explain how the triangulated structures of these two categories are related.

Definition 3.6. *The m -dilatation of a mesh in $\Gamma_{A_{nm-1}}$ ending in (i, j) is given by the mesh of m -diagonals $(i - m, j - m), (i - m, j), (i, j - m)$ and (i, j) , whenever defined.*

Geometrically the m -dilatation enables us to pass from one mesh in Γ , or framed set of diagonals, to one in Γ^m in the following way. One starts with the diagonal (i, j) of Π and rotate it of ρ_m degrees counterclockwise around the center of Π . This gives an m -diagonal $(i - m, j - m)$, which crosses (i, j) by Lemma 3.1. This crossing can be framed in a unique way by the two m -diagonals $(i - m, j)$ and $(i, j - m)$. The set of diagonals we obtain is then the m -dilatation of the mesh of (i, j) .

Proposition 3.7. *The m -dilatation of the mesh in $\Gamma_{A_{nm-1}}$ ending in (i, j) maps the AR-triangle of $M_{(i,j)}$ in \mathcal{C} to the one in \mathcal{C}^m .*

Proof. The AR triangles of $M_{(i,j)}$ in \mathcal{C} , respectively in \mathcal{C}^m were given in Corollary 3.5. The respective framed sets of diagonals are $(i - 1, j - 1), (i, j - 1), (i - 1, j), (i, j)$ and $(i - m, j - m), (i, j - m), (i - m, j), (i, j)$, which are linked by an m -dilatation. \square

3.2. Triangles arising from a morphism in \mathcal{C}^m . We saw that the indecomposable objects of \mathcal{C}^m are isomorphic to certain indecomposable objects of \mathcal{C} , and since both categories are triangulated one has that for a morphism μ from M to N there is a triangle $M \xrightarrow{\mu} N \rightarrow P \rightarrow \Sigma M$ in \mathcal{C} and one in \mathcal{C}^m . In this section we study how those two triangles are related to each other. To answer this we first recall the following definition, which will help understanding the morphisms in $\text{Ind}(\mathcal{C}^m)$ as the latter is equivalent to the mesh category of $\Gamma_{A_{nm-1}}^m$ (Theorem 5.6 in [BaM1]).

Definition 3.8. *The mesh category of a translation quiver Q is the factor category of the path category of Q modulo the ideal generated by the mesh relations*

$$r_v := \sum_{\alpha: u \rightarrow v} \alpha \cdot \sigma(\alpha),$$

where the sum is over all arrows ending in v , and v runs through the vertices of Q .

Geometrically the mesh relations are described in [CCS] as follows. First recall that two diagonals α and α' of Π are related by a *pivoting elementary move* if they share a vertex, called *pivot*, and if there is a positive rotation around the pivot from α to α' inside Π . This rotation must be minimal. A *pivoting path* from α to α' is a sequence of pivoting elementary moves starting at α and ending at α' . If there are m moves in such a path, we say that the path *has length m* . Now, for any two diagonals α and α' of Π such that α is related to α' by two consecutive pivoting elementary moves with distinct pivots, one defines the *mesh relations*: $P_{v'_2} P_{v_1} = P_{v'_1} P_{v_2}$, where v_1, v_2 are the vertices of the diagonal α and v'_1, v'_2 are the vertices of α' , such that $P_{v'_1} P_{v_2}(\alpha) = \alpha'$. In this definition the border edges of Π are allowed with the convention that the term corresponding to such an edge in the mesh relation is replaced by zero.

As next we turn our attention to the morphisms in \mathcal{C}^m , for this we let $A := \bigoplus_{1 \leq i \leq m} M_i$ and $B := \bigoplus_{1 \leq j \leq n} N_j$ be objects in \mathcal{C}^m with M_i and N_j indecomposables for all i, j 's.

Since \mathcal{C}^m is an additive category one can identify an arbitrary $\mu \in \text{Hom}_{\mathcal{C}^m}(A, B)$ with the $m \times n$ matrix $(\mu_{i,j})_{1 \leq i, j \leq m, n}$, where $\mu_{i,j} : \pi_j \circ \mu \circ \iota_i \in \text{Hom}_{\mathcal{C}^m}(M_i, N_j)$, with $\pi_j : B \twoheadrightarrow N_j$, and $\iota_i : M_i \hookrightarrow A$, as can be seen for example in Appendix A in [ASS].

Furthermore, from the shape of the AR quiver of the m -cluster category it follows that elements in $\text{Hom}_{\mathcal{C}^m}(M_i, N_j)$ are compositions of irreducible morphisms, and the latter satisfy the following property.

Lemma 3.9. *Let $M, N \in \text{Ind}(\mathcal{C}^m)$. Then $\mu \in \text{Irr}_{\mathcal{C}^m}(M, N)$ if and only if*

$$\mu = \nu_{m-1} \circ \cdots \circ \nu_0$$

for $I_i \in \mathcal{C}$, $0 \leq i \leq m$, such that $I_0 = M$, $I_m = N$ and $\tau I_{i+1} \not\cong I_{i-1}$ for $0 < i < m$ and $\nu_i \in \text{Irr}_{\mathcal{C}}(I_i, I_{i+1})$ for $0 \leq i \leq m-1$

Proof. One implication is obvious, for the other one let $\mu \in \text{Irr}_{\mathcal{C}^m}(M, N)$, and consider the AR triangle in \mathcal{C}^m

$$M \xrightarrow{(\mu_1, \mu_2)} N_1 \oplus N_2 \rightarrow \tau_m^{-1} M \rightarrow \Sigma M$$

where $\mu_i \in \text{Irr}_{\mathcal{C}^m}(M, N_i)$ for $i = 1, 2$. From the shape of the AR quiver of \mathcal{C}^m and since we assume both that N is indecomposable and μ is irreducible, we deduce that either $N \cong N_1$ or $N \cong N_2$. Otherwise μ would be the zero morphism. Consequently, μ is either equal to μ_1 or to μ_2 . Furthermore, it follows from Theorem 2.1, Proposition 2.3 and the definition of m -th power of a translation quiver that as morphism in \mathcal{C} , μ has to be of the form $\mu = \nu_{m_1} \circ \nu_{2_1} \circ \cdots \circ \nu_{1_1}$ or $\mu = \nu_{m_2} \circ \nu_{2_2} \circ \cdots \circ \nu_{1_2}$, here $\nu_{i_j} \in \text{Irr}_{\mathcal{C}}(I_i, I_{i_j})$ for $j = 1, 2$; $I_0 := M$ and $I_m := N$. This shows that μ corresponds to a sectional path of length m in $\Gamma_{A_{nm-1}}$ starting at the vertex for I_0 and with I_{i_j} corresponding to the vertices along this path. \square

In geometric terms the previous result says that if we associate to the indecomposables $M, N \in \mathcal{C}^m$ the m -diagonals (i, j) and (k, l) in Π then $\mu \in \text{Hom}_{\mathcal{C}^m}(M, N)$ is irreducible if and only if it corresponds to a composition of m elementary moves around a common pivot, modulo the mesh relations.

To give a geometric meaning of the third term in the triangle associated to a morphism we want to study short exact sequences. But since cluster categories are not abelian, not every morphism admits a kernel and cokernel, so we don't have a direct access to short exact sequences. Nevertheless, the next result tells us how to bridge this obstacle.

Remember from Proposition 1.6 in [BMRRT] that for an $M \in \text{Ind}(\mathcal{C})$ a representative of $\pi^{-1}(M)$ under the projection $\pi : \mathcal{D} \rightarrow \mathcal{C}$ is an object contained in the *fundamental domain* \mathcal{S} for the action of $F := \tau^{-1}\Sigma$ on $\text{Ind}(\mathcal{D})$. \mathcal{S} is defined as the union of all indecomposable kQ -modules (viewed as stalk complexes concentrated in degree 0) together with the objects ΣP_j , where the P_j 's are projective indecomposable kQ -modules.

Lemma 3.10. *For $M, N \in \text{Ind}(\mathcal{C})$*

$$\text{Hom}_{\mathcal{C}}(M, N) \cong \text{Hom}_{\text{mod } kQ}(M', N').$$

Before passing to the proof observe that $\text{Hom}_{\mathcal{D}}(M^\bullet, N^\bullet) \cong \text{Hom}_{\text{mod } kQ}(M, N)$, for M^\bullet, N^\bullet indecomposable objects in \mathcal{D} with indecomposable stalks M, N concentrated in the same degree. The above isomorphism is independent of the degree the stalks are concentrated in.

Proof of Lemma 3.10. If the preimages $\widetilde{M}, \widetilde{N} \in \mathcal{S}$ under π of M, N are both in $\text{mod } kQ$ and \widetilde{M} happens to be projective, the result follows from Proposition 1.7 in [BMRRT]. If this is not the case, we need to consider a τ^j -shift of \mathcal{S} , for a certain $j \in \mathbb{Z}$.

In fact, assume that $\widetilde{M} \in \mathcal{S}$ is not a projective indecomposable kQ -module. Then the action of F on the AR-quiver of $\text{Ind}(\mathcal{D})$, which as we pointed out earlier is isomorphic to $\mathbb{Z}A_n$, implies that there are $j, k, l \in \mathbb{Z}$ such that $F^j(\widetilde{M}) \cong \Sigma^k P_l$, i.e. there is a projective indecomposable P_l concentrated in degree k in the F -orbit of \widetilde{M} . Call $\widetilde{M}' := \Sigma^k P_l$ and let $\widetilde{N}' := F^j(\widetilde{N})$ be the corresponding representative

of the F -orbit of \tilde{N} in the new fundamental domain $\mathcal{S}' := F^j(\mathcal{S})$. Then we have to consider two cases. First assume that $\tilde{N}' \not\cong \Sigma^{k+1}P_i$. Then \tilde{M}' and \tilde{N}' are both concentrated in degree k and \tilde{M}' is projective. Thus

$$\mathrm{Hom}_{\mathcal{C}}(M, N) \cong \mathrm{Hom}_{\mathrm{mod}(kQ)}(\tilde{M}', \tilde{N}')$$

again by Proposition 1.7 in [BMRRT] together with the observation previous the proof. Secondly, if $\tilde{N}' \cong \Sigma^{k+1}P_i$

$$\begin{aligned} \mathrm{Hom}_{\mathcal{C}}(M, N) &\cong \mathrm{Hom}_{\mathcal{D}}(F^{-1}\tilde{M}', \tilde{N}') \oplus \mathrm{Hom}_{\mathcal{D}}(\tilde{M}', \tilde{N}') \\ &\cong \mathrm{Hom}_{\mathcal{D}}(\tau\Sigma^{k-1}P_l, \Sigma^{k+1}P_i) \oplus \mathrm{Hom}_{\mathcal{D}}(\Sigma^k P_l, \Sigma^{k+1}P_i) \\ &\cong \mathrm{Hom}_{\mathcal{D}}(\tau P_l, \Sigma^2 P_i) \oplus \mathrm{Hom}_{\mathcal{D}}(P_l, \Sigma P_i) \\ &\cong \mathrm{Hom}_{\mathcal{D}}(\Sigma^{-1}I_{l'}, \Sigma^2 P_i) \oplus \mathrm{Hom}_{\mathcal{D}}(P_l, \Sigma P_i) \\ &\cong \mathrm{Hom}_{\mathcal{D}}(I_{l'}, \Sigma^3 P_i) \oplus \mathrm{Hom}_{\mathcal{D}}(P_l, \Sigma P_i) \\ &\cong \mathrm{Ext}_{\mathrm{mod}kQ}^3(I_{l'}, P_i) \oplus \mathrm{Ext}_{\mathrm{mod}kQ}^1(P_l, P_i) \\ &= 0 \cong \mathrm{Hom}_{\mathrm{mod}kQ}(\tilde{M}', \tilde{N}') \end{aligned}$$

Here the first isomorphism follows from the fact that the category $\mathrm{mod}kQ$ is hereditary, furthermore we used that $\tau P_l \cong \Sigma^{-1}I_{l'}$. \square

Our next goal is to determine the third object in the distinguished triangle

$$M \xrightarrow{\mu} N \rightarrow L \rightarrow \Sigma M \quad \text{in } \mathcal{C}$$

into which the morphism μ embeds.

The notation in the next theorem is as follows: $M_{(i,j)}$, $M_{(k,l)}$ are the indecomposable objects associated to the diagonals (i,j) and (k,l) of Π noticing that here $i < j$ and $k \geq i$.

Theorem 3.11. *Let $\mu \in \mathrm{Hom}_{\mathcal{C}}(M_{(i,j)}, M_{(k,l)})$, then*

$$\mathrm{Cone}(\mu) \cong \begin{cases} M_{(j-1,l)} & \text{if } \mu \text{ is injective,} \\ \Sigma M_{(i,1+k)} & \text{if } \mu \text{ is surjective,} \\ 0 & \text{if } \mu \text{ is an isomorphism,} \\ \Sigma M_{(i,1+k)} \oplus M_{(j-1,l)} & \text{otherwise.} \end{cases}$$

Proof. The strategy of the proof is the following. First lift the morphism μ to \mathcal{D} which by Lemma 3.10 can be done easily in this case. Then complete the morphism to a distinguished triangle in \mathcal{D} and explicitly compute the cone of μ in \mathcal{D} . At the end project the resulting triangle through the triangle functor $\pi : \mathcal{D} \rightarrow \mathcal{C}$ to obtain the distinguished triangle associated to μ in \mathcal{C} . By abuse of notation we write $\mathrm{Cone}(\mu) := \pi(\mathrm{Cone}(\mu))$.

After doing a τ^j shift of the fundamental domain \mathcal{S} for a certain $j \in \mathbb{Z}$ as described in the proof of Remark 3.10 we can assume that $i = 1$. Furthermore, we assume Q to be equally oriented. We proceed considering the different cases that occur for μ . First case: assume $M_{(k,l)} = M_{(1,l)}$ with $l \neq j$, so that μ is injective. Then, consider the short exact sequence of indecomposable modules in the abelian category $\mathrm{mod}kQ$

$$0 \rightarrow M_{(1,j)} \xrightarrow{\mu} M_{(1,l)} \xrightarrow{\nu} \mathrm{coker}(\mu) \cong M_{(1,l)}/M_{(1,j)} \rightarrow 0$$

which by the embedding of $\mathrm{mod}kQ$ into \mathcal{D} can be associated to the distinguished triangle

$$M_{(1,j)} \xrightarrow{\mu} M_{(1,l)} \rightarrow M_{(1,l)}/M_{(1,j)} \xrightarrow{\delta} \Sigma M_{(1,j)}$$

where $\delta = (id, 0) \circ \phi^{-1}$ and $\phi : \text{cone}(\mu) \rightarrow M_{(1,l)}/M_{(1,j)}$ is the map of complexes:

$$\begin{array}{ccccccc} \cdots & \longrightarrow & M_{(1,j)} & \xrightarrow{\mu} & M_{(1,l)} & \longrightarrow & 0 \longrightarrow \cdots \\ & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & 0 & \longrightarrow & M_{(1,l)}/M_{(1,j)} & \longrightarrow & 0 \longrightarrow \cdots \end{array}$$

which is a quasi-isomorphism by Proposition 1.7.5 in [KS]. Notice that δ in \mathcal{D} is non zero since the short exact sequence does not split as $M_{(1,j)}$ is not an injective module ([KS]). Moreover, if one identifies the vertices of the AR quiver of $\text{mod } kQ$ with the positive roots of the root system corresponding to Q one easily sees that $M_{(1,l)}/M_{(1,j)} \cong M_{(j-1,l)}$ in terms of objects associated to diagonals in $\Gamma_{A_{nm-1}}$. One concludes considering that the previous triangle projects to the distinguished triangle of \mathcal{C}

$$M_{(1,j)} \xrightarrow{\mu} M_{(1,l)} \rightarrow \text{Cone}(\mu) \rightarrow \Sigma M_{(1,j)}.$$

So that $\text{Cone}(\mu) \cong M_{(j-1,l)}$ by the triangulated version of the Five Lemma applied in \mathcal{C} .

Second case: assume that the target of μ is $M_{(k,j)}$ and $i < k \leq j-2$. So μ is surjective and one has the short exact sequence

$$0 \rightarrow \ker(\mu) \hookrightarrow M_{(1,j)} \xrightarrow{\mu} M_{(k,j)} \rightarrow 0.$$

This can be embedded into the distinguished triangle in \mathcal{D} :

$$M_{(1,1+k)} \xrightarrow{u} M_{(1,j)} \xrightarrow{\mu} M_{(k,j)} \xrightarrow{\nu} \Sigma M_{(1,1+k)}$$

noticing that $\ker(\mu) \cong M_{(1,1+k)}$, as it follows from the shape of the AR quiver of $\text{mod } kQ$. Since $M_{(k,j)}$ is never projective and $M_{(1,1+k)}$ is never injective, it follows that ν is non zero, as the short exact sequence does not split. By the rotation axiom in a triangulated category \mathcal{D} one has that the upper row in the next diagram is again a distinguished triangle

$$\begin{array}{ccccccc} M_{(1,j)} & \xrightarrow{\mu} & M_{(k,j)} & \xrightarrow{\nu} & \Sigma M_{(1,1+k)} & \xrightarrow{-\Sigma u} & \Sigma M_{(1,j)} \\ \parallel & & \parallel & & \downarrow \exists & & \parallel \\ M_{(1,j)} & \xrightarrow{\mu} & M_{(k,j)} & \longrightarrow & \text{Cone}(\mu) & \longrightarrow & \Sigma M_{(1,j)} \end{array}$$

Hence as in the previous case applying π one obtains that $\text{Cone}(\mu) \cong \Sigma M_{(1,1+k)}$ in \mathcal{C} .

Third case: if μ is an isomorphism it follows from the previous cases that $\text{Cone}(\mu)$ must be isomorphic to the zero object.

Last case: μ is neither surjective nor injective. In this situation it follows from the shape of the AR quiver of $\text{mod } kQ$ that μ can be chosen to be the composition of $f : M_{(1,j)} \hookrightarrow M_{(1,l)}$, followed by $g : M_{(1,l)} \twoheadrightarrow M_{(k,l)}$. Combining the triangles associated to these morphisms obtained in the same way as in the previous cases one gets:

Here $\text{coker}(f) \cong M_{(j-1,l)}$ and $\Sigma \ker(g) \cong \Sigma M_{(1,1+k)}$. By the octahedral axiom one obtains the distinguished triangle

$$M_{(j-1,l)} \rightarrow \text{Cone}(\mu) \rightarrow \Sigma M_{(1,1+k)} \xrightarrow{h} \Sigma M_{(j-1,l)}$$

with $h : \Sigma M_{(1,1+k)} \rightarrow \Sigma M_{(j-1,l)}$ the zero morphism. In fact

$$\text{Hom}_{\mathcal{D}}(\Sigma M_{(1,1+k)}, \Sigma M_{(j-1,l)}) \cong \text{Hom}_{\mathcal{D}}(M_{(1,1+k)}, M_{(j-1,l)})$$

and it can be checked that $M_{(j-1,l)}$ is outside of the support of $\text{Hom}_{\mathcal{D}}(M_{(1,1+k)}, ?)$.

By the rotation axiom applied to the previous triangle one obtains the first row in the next diagram:

$$\begin{array}{ccccccc} M_{(1,1+k)} & \xrightarrow{0} & M_{(j-1,l)} & \longrightarrow & \text{Cone}(\mu) & \longrightarrow & \Sigma M_{(1,1+k)} \\ \parallel & & \parallel & & \downarrow \exists & & \parallel \\ M_{(1,1+k)} & \xrightarrow{0} & M_{(j-1,l)} & \longrightarrow & \Sigma M_{(1,1+k)} \oplus M_{(j-1,l)} & \longrightarrow & \Sigma M_{(1,1+k)} \end{array}$$

Here we used that $\text{Cone}(0) \cong \Sigma M_{(1,1+k)} \oplus M_{(j-1,l)}$. The Five-Lemma then implies that $\text{Cone}(\mu) \cong \Sigma M_{(1,1+k)} \oplus M_{(j-1,l)}$ in \mathcal{D} . One concludes projecting the triangle to \mathcal{C} . \square

In particular, it follows from the next result that taking the m -th power of the AR quiver $\Gamma_{A_{nm-1}}$ of \mathcal{C} doesn't give rise to new triangles between indecomposable objects.

Corollary 3.12. *Let $M, N \in \text{Ind}(\mathcal{C}^m)$ and $\mu \in \text{Hom}_{\mathcal{C}^m}(M, N)$. Then $\text{Cone}(\mu) \in \mathcal{C}^m$ is as described in Theorem 3.11.*

Proof. From Lemma 3.9 one deduces that \mathcal{C}^m is a subcategory of \mathcal{C} , therefore it follows from the Five-Lemma that the triangle associated to μ in \mathcal{C}^m is isomorphic to the corresponding one in \mathcal{C} . \square

4. COMPONENTS OF $((\Gamma_{A_{mn-1}})^m, \tau^m)$

In Section 2 we saw that $(\Gamma_{A_{n-1}}^m, \tau_m)$ is a connected component of the m -th power of $(\Gamma_{A_{mn-1}}, \tau)$. So far it was not clear how to characterize all the remaining connected components arising when taking the m -th power of the translation quiver $(\Gamma_{A_{mn-1}}, \tau)$. In fact, the answer depends on the parity of m . The main difference is that when m is even a further symmetry in the translation quiver $(\Gamma_{A_{mn-1}}, \tau)$ appears, due to the symmetry of the polygon Π around its central diagonals, i.e. those of the form $(i, i + \frac{N}{2})$. This symmetry increases the number of connected components arising.

This section is divided into two parts and in the first one we present two results of the unpublished work of C. Ducrest, [Du], who was able to describe all connected components for odd values of m . In the second part we focus on even m and give a complete characterization of the connected components. Our strategy is to study the action of the automorphisms τ^{-1} and Σ on $\mathbb{Z}A_n$. For this it is convenient to declare the horizontal distance between two neighbored vertices of $\mathbb{Z}A_n$ to be one.

Remark 4.1. *The action of τ^{-1} can be seen as moving each vertex one unit to the right. The action of Σ^m is to move each vertex $m \frac{n+1}{2}$ units to the right followed by a reflection around the horizontal center line whenever m is odd.*

From this it follows that $\mathbb{Z}A_n / \tau^{-1} \Sigma^m$ has the shape of a cylinder when m is even, or of a Möbius band if m is odd.

Definition 4.2. Let (Γ, τ) be $(\Gamma_{A_{mn-1}}, \tau)$ or $(\Gamma_{A_{n-1}}^m, \tau_m)$. We call the i -column of Γ the line in which all the diagonals, or m -diagonals, of the polygon Π containing the vertex i are listed.

Remark that i -columns come with two possible slopes as they can go in two different directions. Furthermore, from now on we write (Γ, τ) for $(\Gamma_{A_{mn-1}}, \tau)$.

The next result characterizes the τ -orbits containing the vertices of Γ in terms of their τ_m -orbits. In particular: we will see that if m is odd, the two orbits always coincide.

Proposition 4.3. The number of connected components of $((\Gamma)^m, \tau^m)$ meeting any τ -orbit of Γ is:

- (1) 1 for m odd,
- (2) 1 or 2 for m even, and if it is 2 then the corresponding connected components are isomorphic.

Furthermore, each connected component of $((\Gamma)^m, \tau^m)$ contains vertices of exactly one of the τ -orbits of $(1, 3), (1, 4), \dots, (1, \lceil \frac{m-1}{2} \rceil + 2)$ or $(1, m)$.

Proof. This follows directly from Lemma 4.2.19 and 4.2.20 in [Du]. \square

Observe that the vertices in the τ -orbit of $(1, km)$, $k \geq 1$, give rise to the connected component of the m -th power of Γ isomorphic to the AR quiver of \mathcal{C}^m , c.f. Theorem 2.1. The next result describes explicitly the connected components into which the m -th power decomposes when m is odd. An illustration of this result is presented in Section 5.

Theorem 4.4. [Du, Thm: 4.2.24] For all $m \geq 1$, there exists a t such that

$$(\Gamma)^m = \Gamma_{A_{n-1}}^m \coprod \bigcup_{i=1}^t \Gamma_i,$$

is the decomposition of $((\Gamma)^m, \tau^m)$ into connected components, and each Γ_i is isomorphic to the Auslander-Reiten quiver of the orbit category of $\mathcal{D}^b(kA_n)$ under an auto-equivalence of the form $\tau^{-s} \circ \Sigma^r$, for some r, s with $s < n$. If m is odd then $t = \frac{m-1}{2}$, $r = m+1$ and $s = \frac{m-1}{2}n - \frac{m-3}{2}$, i.e.

$$(\Gamma)^m = \Gamma_{A_{n-1}}^m \coprod \bigcup_{i=1}^{\frac{m-1}{2}} \mathbb{Z}A_n / (\tau^{-\frac{m-1}{2}n + \frac{m-3}{2}} \circ \Sigma^{m+1}).$$

The explicit characterization of the auto-equivalence in the previous result yields the following:

Corollary 4.5. Let m be odd, then the connected components Γ_i have all a cylindrical shape, whereas $\Gamma_{A_{n-1}}^m$ lies on a Möbius band.

Proof. Observe that the shift functor Σ in the auto-equivalence: $\tau^{-\frac{m-1}{2}n + \frac{m-3}{2}} \circ \Sigma^{m+1}$ is applied an even number of times. \square

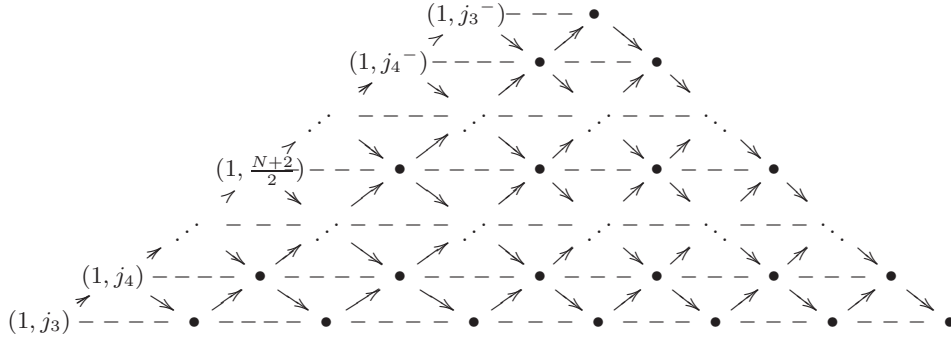
Later on we will see that when m is even this is no longer always true. In fact, a number of orbit categories arising through $(\Gamma)^m$ will have a Möbius band as their Auslander-Reiten quiver.

Proposition 4.6. Let Γ_i be as in Theorem 4.4 and m odd. If $(n+1)$ divides $2(nm+1)$, then $\Gamma_i \cong \Gamma_{A_n}^u$ with $u := \frac{2(nm+1)}{n+1}$.

Proof. By Proposition 1.3 in [BMRRT] we know that the Auslander-Reiten quiver of the u -cluster category of type A_n is isomorphic to $\mathbb{Z}A_n/\tau^{-1} \circ \Sigma^u$. Set $\varphi := \tau^{-s} \circ \Sigma^{m+1}$ where $s = \frac{m-1}{2}n - \frac{m-3}{2}$ by Theorem 4.4 and $\varphi' := \tau^{-1} \circ \Sigma^u$. Then compare their actions on $\mathbb{Z}A_n$ which is isomorphic to the Auslander-Reiten quiver of $\mathcal{D}^b(kA_n)$ by [Ha2]. For u satisfying the assumptions we have that the vertices of $\mathbb{Z}A_n$ have the same orbit under the action of the two auto-equivalences φ and φ' . In fact, each vertex is moved by $(m+1)\frac{n+1}{2} + (\frac{m-1}{2}n - \frac{m-3}{2}) = u\frac{n+1}{2} + 1$ units to the right. Furthermore, since u is even, $\Gamma_i \cong \mathbb{Z}A_n/\varphi$ and $\mathbb{Z}A_n/\varphi'$ are both cylindrically shaped. Thus $\mathbb{Z}Q/\varphi \cong \mathbb{Z}Q/\varphi'$. \square

Example 4.7. Let $m = 5$ and $n = 3$, and consider the connected components of $(\Gamma_{A_{14}})^5$. There are three of them: one is $\Gamma_{A_2}^5$, and the other two are isomorphic to $\Gamma_{A_3}^8$, which corresponds to the Auslander-Reiten quiver of the 8-cluster category of type A_3 .

4.1. The case where m is even. From the assumption on m it follows that the N -gon Π has an even number of sides. The most important difference to the case of odd m is that the polygon contains central diagonals $(i, i + \frac{N}{2})$. Let us consider the diagonals incident with the vertex i . Apart from the central diagonal $(i, i + \frac{N}{2})$ they come in pairs (i, j) and $(i, N+1+i-j)$, symmetric about $(i, i + \frac{N}{2})$. We will use $j^- := N+2-j$ and write (i, j^-) for the diagonal which is the mirror of (i, j) . Observe that the τ -orbit containing the central diagonals forms the middle row of (Γ, τ) .



Remark 4.8. The τ -orbit of $(1, j)$ also contains $(1, j^-)$ and the cardinality of it is N unless $j = j^-$. If so, the cardinality is $\frac{N}{2}$.

Furthermore, it follows from the definition of τ^m that by distinguishing the endpoints of a given diagonal (i, j) according to their parity one has:

- $\tau^m(\text{even}, \text{even}) = (\text{even}, \text{even})$
- $\tau^m(\text{odd}, \text{odd}) = (\text{odd}, \text{odd})$
- $\tau^m(\text{even}, \text{odd}), \tau^m(\text{odd}, \text{even}) \subset \{(\text{odd}, \text{even}), (\text{even}, \text{odd})\}$.

We will refer to these properties as *parity configurations*.

The next results sharpen Proposition 4.3.

Lemma 4.9. Let (i, j) be a vertex of Π . Assume that m and $|i - j|$ are even. Then the τ -orbit of (i, j) meets two components of $((\Gamma)^m, \tau^m)$.

Proof. Without loss of generality we can assume $i = 1$.

Since j is odd it follows from the parity configurations that all the elements of the τ^m -orbit containing (i, j) are diagonals whose end points are both odd. Hence, diagonals with vertices of the form $(\text{even}, \text{even})$ never belong to the τ^m -orbit of

$(1, j)$. Therefore, there is strictly more than one component in the τ -orbit of Γ through $(1, j)$. From Proposition 4.3 it follows that there must be exactly two. \square

Corollary 4.10. *Let m and $|i - j|$ be even, then the cardinality of the τ^m -orbit containing (i, j) is $\frac{N}{2}$.*

Proof. Follows from Lemma 4.9 and Remark 4.8. \square

It remains to consider τ -orbits of vertices (i, j) of (Γ, τ) for odd $|i - j|$. In the proof of the next lemma it becomes clear how the symmetry of the polygon interferes with the connected components.

Lemma 4.11. *Let m be even and $|i - j|$ be odd. Then, the τ -orbit of (i, j) meets only one connected component of $((\Gamma)^m, \tau^m)$ if and only if (i, j) and (i, j^-) belong to the same connected component.*

Proof. As before w.l.o.g. let $i = 1$ and observe that the τ^m -orbit containing $(1, j)$ and the τ^m -orbit of $(1, j^-)$ are disjoint unless $j = j^-$, i.e. $j = \frac{N+2}{2}$. In fact, modulo N the τ^m -orbit of $(1, j)$ consists of diagonals of the form $(1 - rm, j - rm)$. In particular, all its elements have parity configuration: (odd, even), or (even, odd). Therefore, we only have to show that the τ^m -orbit of $(1, j)$ does not contain $(2, j+1)$ exactly when $(1, j^-) \notin \tau^m(1, j)$. Assume instead that there is a $k_0 \in \mathbb{N}$ such that both $1 - k_0m \equiv j + 1$ and $j - k_0m \equiv 2$ modulo N . Then N would divide their difference: $|(j + 1 - 2) - (1 - k_0m - j + k_0m)| = |2j - 2|$. But this can only happen if $(1, j)$ is a central diagonal, i.e. $j = j^-$, which is equivalent to $j = \frac{N+2}{2}$. Thus, we showed that

$$(2, j + 1) \in \tau^m(1, j) \iff (1, j^-) \in \tau^m(1, j).$$

To conclude recall that the τ -orbit of $(1, j)$ coincides with the one of $(1, j^-)$. \square

Corollary 4.12. *Let m be even and $|i - j|$ be odd, then the cardinality of the τ^m -orbit of (i, j) is $\frac{N}{2}$.*

Proof. In fact, when $|i - j|$ is odd the τ^m -orbit of (i, j) and the τ^m -orbit of (i, j^-) are disjoint if $j \neq j^-$, as we observed in the proof of Lemma 4.11. These two sets together give the τ orbit of (i, j) . If $j = j^-$ then the τ and τ^m orbit coincide. Hence, we conclude with Remark 4.8. \square

From now on we denote by $\Gamma_{(i, j)}$ the connected component of $((\Gamma)^m, \tau^m)$ containing the diagonal (i, j) .

Corollary 4.13. *For $m = 2$, $((\Gamma)^2, \tau^2)$ splits into three connected components*

$$(\Gamma)^2 = \Gamma_{A_{n-1}}^2 \cup \Gamma_{(1,3)} \cup \Gamma_{(2,4)}.$$

Proof. For $m = 2$, it is clear that for every even j , $(1, j)$ and $(1, j^-)$ always belong to the same connected component. Hence, by Lemma 4.11 the τ -orbit of $(1, 2k)$, $k \in \mathbb{N}$, meets one connected component of $((\Gamma)^m, \tau^m)$ and the τ -orbit of $(1, 2k + 1)$, $k \in \mathbb{N}$ meets two by Lemma 4.9. \square

Observe that with Proposition 4.3 we were only able to conclude that the second power of Γ consists of two or three connected components.

Next we give a criterion to determine whether the mirror diagonal (i, j^-) of (i, j) , for $j \neq j^-$, is or is not part of the connected component containing (i, j) .

Lemma 4.14. *Let m be even, then (i, j^-) and (i, j) belong to the same connected component of $((\Gamma)^m, \tau^m)$ if and only if m divides $|2(2 - j)|$.*

Proof. Let $i = 1$. The previous Lemma 4.11 the τ -orbit of $(1, j)$ meets one connected component if and only if the vertices $(1, j)$ and $(1, j^-)$, where $j^- = N + 2 - j$, are part of the same connected component of $((\Gamma)^m, \tau^m)$. This is equivalent to saying that there is a $r_0 \in \mathbb{N}$ such that $(n - r_0)m = 4 - 2j$. \square

Proposition 4.15. *For $m > 2$ even and $|i - j|$ odd the τ -orbit of (i, j) always meets two connected components except if $|i - j| = \frac{m}{2} + 1$ then it only meets one connected component.*

Proof. Follows from Lemmas 4.9, 4.11 and 4.14. In fact, $m \mid |2(2 - j)|$ if and only if $j = \frac{m+4}{2}$, since for all other values of j one has that $m > |2(2 - j)|$. \square

Combining the previous results we are now able to state the main result of this section, which completes the description of the connected components of the m -th power of Γ .

Theorem 4.16. *Let m be even. Then $((\Gamma)^m, \tau^m)$ decomposes into m or $m + 1$ connected components. In fact*

$$\begin{aligned} (\Gamma)^m &= \Gamma_{A_{n-1}}^m \cup \bigcup_{i=1}^{\frac{m}{2}-1} \left(\Gamma_{(1,i+2)} \cup \Gamma_{(2,i+3)} \right) \cup \Gamma_{(1, \frac{m}{2}+2)}, & \text{if } \frac{m}{2} \text{ is even;} \\ (\Gamma)^m &= \Gamma_{A_{n-1}}^m \cup \bigcup_{i=1}^{\frac{m}{2}-1} \left(\Gamma_{(1,i+2)} \cup \Gamma_{(2,i+3)} \right) \cup \bigcup_{k=1}^2 \Gamma_{(k, \frac{m}{2}+1+k)}, & \text{if } \frac{m}{2} \text{ is odd.} \end{aligned}$$

Furthermore, all the connected components aside from $\Gamma_{A_{n-1}}^m$ are isomorphic to $\mathbb{Z}A_n / \tau^{-\frac{n(m-2)}{2}} \Sigma^2$ and cylindrically shaped except $\Gamma_{(1, \frac{m}{2}+2)}$ and $\Gamma_{(2, \frac{m}{2}+3)}$ which are isomorphic to $\mathbb{Z}A_n / \tau^{-\frac{n(m-2)}{4}} \Sigma$ and have the shape of a Möbius band.

Proof. To begin recall that by Proposition 4.3 we know that each connected component different from $\Gamma_{A_{n-1}}^m$ has a vertex in exactly one τ -orbit of the form $\tau(1, j+2)$, for $j \in \{1, \dots, \lceil \frac{m-1}{2} \rceil = \frac{m}{2}\}$. By Lemma 4.9 and Proposition 4.15 we know that all the τ -orbits for $j < \frac{m}{2}$ meet two isomorphic connected components of the m -th power of Γ , and that $\{\tau^k(1, \frac{m}{2}+2) \mid k \geq 1\}$ meets one or two depending on the parity of j . Hence the number of connected components in the decomposition is clear.

To concretely describe the automorphisms that give rise to the different connected components set $j_0 = \frac{m}{2} + 2$, and as the shape of the connected components depends on the parity of j_0 one proceeds by cases.

First assume j_0 is odd. By Lemma 4.9 we know that the τ -orbit of $(1, j_0)$ in Γ meets two connected components which are isomorphic by Proposition 4.3. We now show that they have the shape of a Möbius band. In fact, by Lemma 4.14 and by the definition of the m -th power of Γ one has that $(1, j_0)$ and $(1, j_0^-)$ belong to the 1-column of $\Gamma_{(1, j_0)}$. Moreover, since the τ^m -orbit of $(1, j_0)$ and $(1, j_0^-)$ coincide, each connected component is a Möbius band.

If j_0 is even, we know by Proposition 4.15 that the τ -orbit of $(1, j_0)$ in Γ meets only one connected component. Furthermore, by the same argument as in the previous case one has that $(1, j_0)$ and $(1, j_0^-)$ are vertices of the 1-column of $\Gamma_{(1, j_0)}$. But from the proof of Lemma 4.11 it is clear that the τ^m -orbit of $(1, j_0)$ does not coincide with the τ^m -orbit of $(1, j_0^-)$. Hence, $(1, j_0)$ and $(1, j_0^-)$ cannot be identified, and therefore the shape of the component is the one of a cylinder.

Now, we only have to check what happens in the τ -orbits of $(1, j)$ if $j < \frac{m}{2} + 2$. In this case one deduces from Lemma 4.14 that the mirror diagonal $(1, j^-)$ is never a vertex of the 1-column of $\Gamma_{(1, j)}$. Thus, as before we conclude that the components have the shape of a cylinder.

To conclude we only need to determine the action of the auto-equivalences in terms of powers of the AR translate τ and the shift Σ . To do so we first observe that such an expression is not unique. For our purpose it is convenient to choose the one with a minimal power of Σ . And to determine the appropriate powers of τ one observes that by knowing Γ one easily determines the length of the cylindrically shaped connected components. In fact it is $\frac{N}{2}$ by Corollaries 4.10 and 4.12.

For the Möbius bands one observes that the τ^m -orbits have $\frac{N}{4}$ elements (except for the τ^m -orbit of the middle row, in the case where n is odd). This follows from the fact that the union of the vertices of the two Möbius bands together form a cylindrically shaped component of length $\frac{N}{2}$ and the τ^m -orbits of size $\frac{N}{2}$ meet two τ^m -orbits of such a cylinder.

To express these lengths in terms of Σ and τ one uses Remark 4.1. \square

One notices that all the connected components of the m -th power of (Γ, τ) can be viewed as orbit categories which are triangulated. In fact, if one denotes the induced auto-equivalences $\tau^{-\frac{n(m-2)}{2}}\Sigma^2$, $\tau^{-\frac{n(m-2)}{4}}\Sigma$ and $\tau^{-\frac{n(m-1)+(m-3)}{2}}\Sigma^{m+1}$ on \mathcal{D} by G_1, G_2 and respectively by G_3 , then G_1, G_2 and G_3 are triangle functors satisfying the conditions of Theorem 1 in [Kel].

Hence, it follows that the projection functors $\pi_s : \mathcal{D} \rightarrow \mathcal{D}/G_i$ are triangle functors for $i = 1, 2, 3$. Thus, one has

Corollary 4.17. *For all $m \in \mathbb{N}$, the connected components of $(\Gamma)^m$ are isomorphic to AR quivers of triangulated orbit categories.*

Furthermore, also for even values of m it happens that some of the connected components arising while taking the m -power of (Γ, τ) are isomorphic to AR quivers of u -cluster categories for appropriate values of u . Compare with Lemma 4.6. In fact according to the different cases treated in Theorem 4.16 one can deduce the following.

Proposition 4.18. *Let m be even.*

- (1) *The cylindrically shaped components $\Gamma_{(i,j)}$ of $(\Gamma)^m$ are isomorphic to the AR quiver of an u -cluster category whenever $u := \frac{nm}{n+1}$ is even.*
- (2) *The other $\Gamma_{(i,j)}$'s are isomorphic to the AR quiver of an u -cluster category whenever $u := \frac{nm-2}{2(n+1)}$ is odd.*

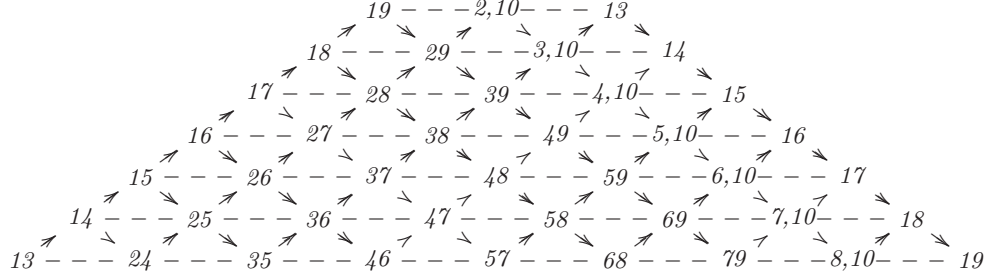
Proof. The proof is analogous as the one of Proposition 4.6 and therefore omitted. \square

One observes that the results of Section 3 apply to all orbit categories arising from the connected components of the m -th power of Γ . Hence they provide a geometric understanding of the triangulated structure of these categories.

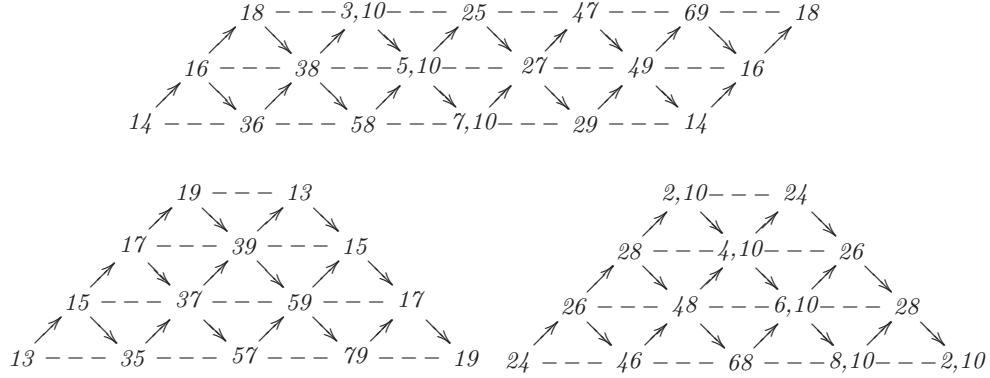
5. APPLICATION

In this section we illustrate on two examples the results presented in Section 4. In particular we illustrate how the shape of the connected components described in Theorem 4.16 vary.

Example 5.1. Let $m = 2$ and $n = 4$. Consider the quiver Γ_{A_7} associated to the diagonals of a decagon:

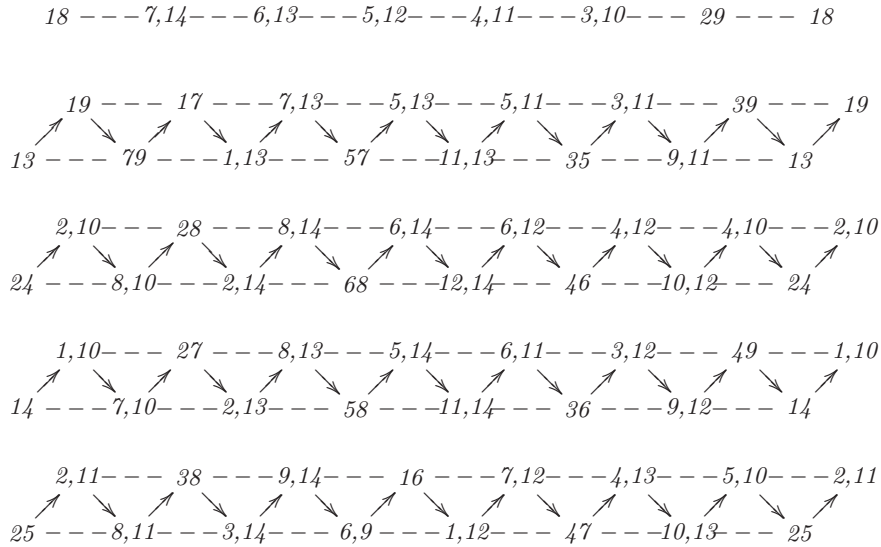


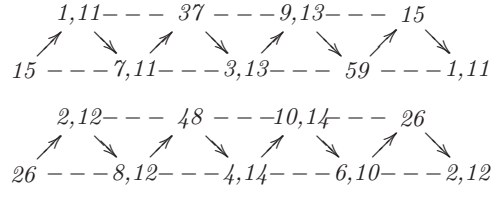
The second power of (Γ_{A_7}, τ) gives rise to three components: $(\Gamma_{A_3}^2, \tau_2)$, $(\Gamma_{(1,3)}, \tau^2) \cong (\Gamma_{(2,4)}, \tau^2) \cong (\mathbb{Z}A_4/\Sigma, \tau^2)$.



By Proposition 4.18 one deduces that there is no $u \in \mathbb{N}$ such that the two connected components isomorphic to $\mathbb{Z}A_4/\Sigma$ are AR quivers of a u -cluster category of type A_4 .

Example 5.2. Let $m = 6$ and $n = 2$. Then the connected components of the 6th-power of $\Gamma_{A_{11}}$ are $\Gamma_{A_1}^6$, and the ones meeting the first three rows of $\Gamma_{A_{11}}$. These are $\Gamma_{(1,3)} \cong \Gamma_{(2,4)} \cong \Gamma_{(1,4)} \cong \Gamma_{(2,5)} \cong \mathbb{Z}A_2/\tau^{-4}\Sigma^2$, which are cylindrically shaped and $\Gamma_{(1,5)} \cong \Gamma_{(2,6)} \cong \mathbb{Z}A_2/\tau^{-2}\Sigma$ having the shape of a Möbius band.





Furthermore, we deduce from Proposition 4.18 that $\Gamma_{(1,3)}$, $\Gamma_{(2,4)}$ and $\Gamma_{(1,4)}$, $\Gamma_{(2,5)}$ are isomorphic to the AR quiver of a 4-cluster category of type A_2 .

REFERENCES

- [Am] C. Amiot, *On the stable structure of triangulated categories with finitely many indecomposables*, Bull. Soc. Math. France **135** (2007), 435-474
- [ASS] I. Assem, D. Simson, A. Skowronski, *Elements of the representation theory of associative algebras*, 1: Techniques of Representation Theory, London Mathematical Society Student Texts 65, Cambridge University Press 2006
- [BaM1] K. Baur, R. Marsh, *A geometric description of m -cluster categories*, Trans. Amer. Math. Soc. **360** (2008), no. 11, 5789-5803
- [BaM2] K. Baur, R. Marsh, *A geometric description of the m -cluster categories of type D_n* , International Mathematical Research Notices (2007), Vol. 2007 doi:10.1093/imrn/rnm011.
- [BaM3] K. Baur, R. Marsh, *A geometric model of tube categories*, arXiv:1011.0743v1
- [BuM] A. B. Buan, R. Marsh, *Cluster-tilting theory*, Trends in representation theory of algebras and related topics, Contemp. Math, Vol. 406, Amer. Math. Soc., Providence, RI, 2006, pp. 1-30.
- [BMRRT] A. B. Buan, R. J. Marsh, M. Reineke, I. Reiten and G. Todorov, *Tilting theory and cluster combinatorics*, Adv. Math. **204** (2006), no. 2, 572-618
- [Bü] Theo Bühler, *An introduction to the derived category*, November 2007, available at: <http://www.uni-math.gwdg.de/theo/>
- [BZ] T. Brüstle, J. Zhang, *On the cluster category of a marked surface*, arXiv:1005.2422v2
- [CCS] P. Caldero, F. Chapoton, and R. Schiffler, *Quivers with relations arising from clusters (A_n -case)*, Trans. Amer. Math. Soc. **358** (2006), no. 3, 1347-1364
- [Du] C. Ducrest, *Powers of translation quivers*, available at <http://www.math.ethz.ch/~baur/Ducrest08.pdf>
- [FZ] S. Fomin, A. Zelevinsky, *Cluster algebras I: Foundations*, J. Amer. Math. Soc. **15** (2002), no. 2, 497-529.
- [Ha1] Dieter Happel, *On the derived category of a finite-dimensional algebra*, Comment. Math. Helv. **62** (1987), no. 3, 339-389.
- [Ha2] Dieter Happel, *Triangulated categories in the representation theory of finite dimensional algebras*, LMS Lecture Note Series 119, Cambridge University Press, 1988
- [HJ] T. Holm, P. Jørgensen, *Triangulated categories: definitions, properties and examples*, available at: www.iazd.uni-hannover.de/~tholm
- [KS] M. Kashiwara, P. Schapira, *Sheaves on Manifolds*, Grundlehren der mathematischen Wissenschaften 292, Springer-Verlag, 1980
- [Ke1] Bernhard Keller, *On triangulated orbit categories*, Documenta Mathematica, Vol. 10 (2005), 551-581.
- [Ke2] Bernhard Keller, *Cluster algebras, quiver representations and triangulated categories*, arXiv:0807.1960
- [Jø] P. Jørgensen, *Quotients of cluster categories*, Proc. Roy. Soc. Edinburgh Sect. A **140** (2010), 65-81
- [Ne] Amnon Neeman, *Triangulated Categories*, Annals of Mathematics Studies, no. 148, Princeton University Press 2001
- [Ri] Christine Riedtmann, *Algebren, Darstellungsköcher, Überlagerungen und zurück*, Comm. Math. Helv. **55**, no. 2 (1990), 199-224.
- [RV] I. Reiten, M. Van den Bergh, *Noetherian hereditary abelian categories satisfying Serre duality*, J. Amer. Math. Soc. **15** (2002) 295-366
- [Sc] Ralf Schiffler, *A geometric model for cluster categories of type D_n* , J. Alg. Comb **27** (2008), no. 1, 1-21
- [Th] H. Thomas, *Defining an m -cluster category*, J. Algebra **318** (1), (2007), 37-46
- [To] H. A. Torkildsen, *A geometric realization of the m -cluster category of type \tilde{A}* , in Enumeration of mutation classes, Doctoral theses of H. A. Torkildsen at NTNU, Trondheim, 2010
- [Wr] A. Wraalsen, *Rigid objects in higher cluster categories*, J. Algebra **321** (2), (2009), 532-547
- [We] Charles A. Weibel, *An introduction to homological algebra*, Cambridge studies in advanced mathematics 38, Cambridge University Press, 1994
- [Zh] B. Zhu, *Generalized cluster complexes via quiver representations*, J. Algebraic Combin. **27** (2008), 24-54
- [ZZ] Y. Zhou, B. Zhu, *Cluster combinatorics of d -cluster categories*, J. Algebra **321** (10) (2009), 2898-2915

